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Photon propagation in a plane-wave field

Ian Affleck†

Physics Department, University of British Columbia, Vancouver, BC, Canada, V6T 2A6

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Abstract. Experiments have been proposed to observe non-linear QED effects using intense plane-wave fields. An external photon propagating in the field develops an index of refraction and a reversed polarisation component. We calculate simple analytic expressions for these effects in the low-energy low-intensity approximation and compare with the constant crossed-field limit and with numerical results.

Although calculations of non-linear QED effects are as old as QED itself, it has only recently become feasible to observe some of these effects experimentally. The proposed experiments [1] would involve high-energy electrons (presumably produced at the Stanford Linear Collider) colliding with an intense focused pulsed optical laser. A Lorentz boost factor of about 10^6 in the centre of mass frame would raise the electric field intensity, of about $10^{13}\ \mathrm{V\,m^{-1}}$, to the threshold for non-linear effects, $eE/m^2 \approx 1$. First round experiments would study non-linear Compton scattering involving coherent adsorbtion and emission of laser photons. An increase in laser intensity by two or three orders of magnitude would allow the observation of strong field effects on the *propagation* of external photons. These effects are weaker because they require the production of a virtual e^+e^- pair which must interact with the plane-wave photons (see figure 1). An external photon develops an index of refraction which may be observable via Cerenkov radiation [2] emitted by electrons travelling through the laser beam. An external photon can also adsorb or emit a pair of laser photons, switching its polarisation.

This index of refraction was calculated analytically for constant crossed electric and magnetic fields [3] (and also for a constant magnetic field [4]). The analogous calculation for a plane-wave field is considerably more complicated. The photon propagator to first order in α but to all orders in eE/m^2, is given by the bubble graph of figure 1. Here, the bold curve represents the exact electron propagator in the

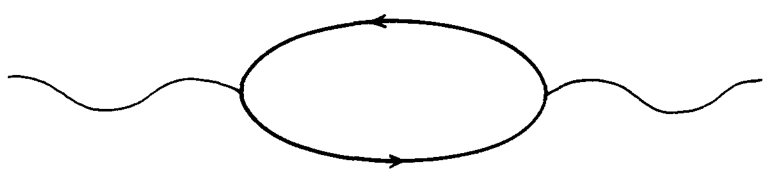

Figure 1. The photon propagator to lowest non-trivial order in α but to all orders in eE. The bold lines represent the exact electron propagator in the background plane-wave field.

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background plane-wave field. The exact electron propagator is known but, nonetheless, the calculation of this graph is extremely laborious [5] and requires numerical integration. It is the purpose of this paper to give simple analytic formulae, valid in the low-intensity low-frequency limit. Comparison with the numerical results [5] indicates that the approximation is good even for intensities and frequencies of order one (compared to the electron mass). Apart from the index of refraction, we also calculate explicitly the polarisation states, give the polarisation flip amplitude, and examine the crossover from the constant field limit at low frequencies to a high-frequency limit in which the external photon is only sensitive to the average squared laser field. This latter limit would be realised experimentally.

The low-intensity and low-frequency approximation to this problem can be derived from the Euler–Heisenberg Lagrangian:

$$L=-\tfrac{1}{4}F^2+(\alpha^2/45m^4)[\tfrac{1}{2}(F^2)^2+\tfrac{7}{8}(F\cdot\tilde{F})^2]$$

where $F^2\equiv F^{\mu\nu}F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ is the dual field. The electromagnetic field, $F_{\mu\nu}$, is composed of the laser and external parts:

$$F=F_{\mathrm{L}}+F_{\mathrm{ex}}.$$

To study the propagation of (low-intensity) external light through the laser field we may expand L to second order in F_{ex} or, equivalently, study the modified Maxwell equations which are linear in F_{ex} (but not F_{L}):

$$L=-\tfrac{1}{4}F_{\mathrm{ex}}^2+(2\alpha^2/45m^4)$$
$$\times[\tfrac{1}{2}F_{\mathrm{L}}^2F_{\mathrm{ex}}^2+(F_{\mathrm{L}}\cdot F_{\mathrm{ex}})^2+\tfrac{7}{8}(F_{\mathrm{L}}\cdot\tilde{F}_{\mathrm{L}})(F_{\mathrm{ex}}\cdot\tilde{F}_{\mathrm{ex}})+\tfrac{7}{4}(F_{\mathrm{L}}\cdot\tilde{F}_{\mathrm{ex}})^2].$$

Using the fact that $F_{\mathrm{L}}^{\mu\nu}\partial_\mu F_{\mathrm{L}\lambda\rho}\propto p_\mu F_{\mathrm{L}}^{\mu\nu}=0$ for a plane wave:

$$F_{\mathrm{L}\mu\nu}=\exp(-\mathrm{i}p\cdot x)f_{\mu\nu}+\mathrm{cc} \tag{1}$$

where $f_{\mu\nu}$ is constant and cc denotes complex conjugate, we obtain the modified Maxwell's equations:

$$\partial_\mu F_{\mathrm{ex}}^{\mu\nu}+2\rho F_{\mathrm{L}}^{\nu\mu}F_{\mathrm{L}}\cdot\partial_\mu F_{\mathrm{ex}}+\tfrac{7}{2}\rho\tilde{F}_{\mathrm{L}}^{\nu\mu}\tilde{F}_{\mathrm{L}}\cdot\partial_\mu F_{\mathrm{ex}}=0 \tag{2a}$$

where $\rho=(\alpha/45\pi)(e/m^2)^2$. (We follow, more or less, the notation and units of [4]; in particular $\alpha=e^2/4\pi\approx 1/137$.)

We consider first the case of a circularly polarised laser field, propagating in the z direction, for which:

$$f_{0i}\equiv e_i \qquad \boldsymbol{e}=E_0(1,\mathrm{i},0)/2$$

$$f_{ij}\equiv\varepsilon_{ijk}b_k \qquad \boldsymbol{b}=E_0(-\mathrm{i},1,0)/2.$$

Here e_i and b_i are the complex electric and magnetic fields and E_0 is their amplitude. Since $b_i=-\mathrm{i}e_i$ for circular polarisation, it follows that the dual field amplitude is simply related to the field amplitude by

$$\tilde{f}_{\mu\nu}=-\mathrm{i}f_{\mu\nu}$$

and equation (2) reduces to

$$\partial_\mu F_{\mathrm{ex}}^{\mu\nu}+\rho[\tfrac{11}{2}(f^{\nu\mu}f^{*\lambda\rho}+f^{*\nu\mu}f^{\lambda\rho})-\tfrac{3}{2}f^{\nu\mu}f^{\lambda\rho}\exp(-\mathrm{i}2p\cdot x)+\mathrm{cc}]\partial_\mu F_{\mathrm{ex}\lambda\rho}=0. \tag{2b}$$

We will find an exact solution of this equation of the form

$$F_{\mathrm{ex}\mu\nu}=f_{\mu\nu}^0\exp(-\mathrm{i}k\cdot x)+f_{\mu\nu}^-\exp[-\mathrm{i}(k-2p)\cdot x]+f_{\mu\nu}^+\exp[-\mathrm{i}(k+2p)\cdot x]+\mathrm{cc}$$

where f^0 represents the incoming external photon and $f^{\pm}$ represent processes in which the external photon emits or adsorbs two laser photons. We express the f^a in terms of polarisation vectors ε^a and momenta

$$f^0_{\mu\nu} = \varepsilon^0_\mu k_\nu - \varepsilon^0_\nu k_\mu$$

and similarly for $f^{\pm}$ with k replaced by $(k \pm 2p)$. We choose the gauge, $\partial_\mu A^\mu = 0$. Equation (2b) then reduces to five complex algebraic equations upon collecting the coefficients of $\exp(-ik \cdot x)$, $\exp[-i(k \pm 2p) \cdot x]$, $\exp[-i(k \pm 4p) \cdot x]$:

$$k^2 \varepsilon^{0\mu} + 11\rho k_\nu (f^{\nu\mu} \varepsilon^0 \cdot f^* \cdot k + f^{*\nu\mu} \varepsilon^0 \cdot f \cdot k)$$
$$-3\rho k_\nu (f^{\nu\mu} \varepsilon^- \cdot f \cdot k + f^{*\nu\mu} \varepsilon^+ \cdot f^* \cdot k) = 0 \tag{3}$$

$$(k \pm 2p)^2 \varepsilon^{\pm\mu} + 11\rho k_\nu (f^{\nu\mu} \varepsilon^{\pm} \cdot f^* \cdot k + f^{*\nu\mu} \varepsilon^{\pm} \cdot f \cdot k) - 3\rho k_\nu \begin{pmatrix} f^{\nu\mu} \varepsilon^0 \cdot f \cdot k \\ f^{*\nu\mu} \varepsilon^0 \cdot f^* \cdot k \end{pmatrix} = 0 \tag{4}$$

$$k_\nu \begin{pmatrix} f^{\nu\mu} \varepsilon^+ \cdot f \cdot k \\ f^{*\nu\mu} \varepsilon^- \cdot f^* \cdot k \end{pmatrix} = 0. \tag{5}$$

Here we have introduced the shorthand

$$\varepsilon \cdot f \cdot k \equiv \varepsilon^\mu f_{\mu\nu} k^\nu \qquad k \cdot f \cdot f \cdot k = k_\mu f^{\mu\nu} f_{\nu\lambda} k^\lambda$$

and used $p^\mu f_{\mu\nu} = 0$. The upper or lower expression in the brackets goes with the + or − sign, respectively. Note that

$$k \cdot F \cdot F \cdot k = k_0^2 \boldsymbol{E}^2 + \boldsymbol{k}^2 \boldsymbol{B}^2 - 2k_0 \boldsymbol{k} \cdot (\boldsymbol{E} \times \boldsymbol{B}) - (\boldsymbol{k} \cdot \boldsymbol{E})^2 - (\boldsymbol{k} \cdot \boldsymbol{B})^2. \tag{6}$$

For a plane wave, travelling in the z direction, this reduces to

$$k \cdot F \cdot F \cdot k = \boldsymbol{E}^2 [k_0^2 + \boldsymbol{k}^2 - 2k_0 k_z - k_x^2 - k_y^2] = \boldsymbol{E}^2 (k_0 - k_z)^2. \tag{7}$$

For circular polarisation, expressing F in terms of the complex amplitude, f, we find

$$k \cdot F \cdot F \cdot k = 2k \cdot f \cdot f^* \cdot k + [k \cdot f \cdot f \cdot k \exp(-2ip \cdot x) + \mathrm{cc}].$$

Since $k \cdot F \cdot F \cdot k$ is constant, this implies

$$k \cdot f \cdot f \cdot k = 0 \tag{8a}$$

$$k \cdot f \cdot f^* \cdot k = E_0^2 (k_0 - k_z)^2 / 2. \tag{8b}$$

We can find solutions of equations (3)-(5) by choosing

$$\varepsilon^{0\mu} = k_\nu f^{\nu\mu} \qquad \varepsilon^{-\mu} = A^- k_\nu f^{*\nu\mu} \qquad \varepsilon^{+\mu} = 0 \tag{9a}$$

or

$$\varepsilon^{0\mu} = k_\nu f^{*\nu\mu} \qquad \varepsilon^{+\mu} = A^+ k_\nu f^{\nu\mu} \qquad \varepsilon^{-\mu} = 0 \tag{9b}$$

where $A^{\pm}$ are constants to be determined. Equation (5) is satisfied identically, due to equation (8a). Equation (4) determines the amplitude $A^{\pm}$:

$$[(k \pm 2p)^2 + 11\rho k \cdot f \cdot f^* \cdot k] A^{\pm} = 3\rho k \cdot f \cdot f \cdot k \tag{10}$$

and equation (3) determines the index of refraction:

$$[k^2 + 11\rho k \cdot f \cdot f^* \cdot k][(k \pm 2p)^2 11\rho k \cdot f \cdot f^* \cdot k] = (3\rho k \cdot f \cdot f^* \cdot k)^2. \tag{11}$$

These two different solutions are in fact the same, with k translated by $2p$. In both cases there are two Fourier modes, with momenta differing by $2p$. In general we expect two independent solutions of equations ($2b$), or of equations (10) and (11) with a *fixed* sign, corresponding to two different polarisations.

We now wish to consider two limits of the above results. The first is the limit $p \to 0$, or more accurately,

$$k \cdot p \ll \rho k \cdot f \cdot f^* \cdot k.$$

The plane wave then reduces to constant crossed electric and magnetic fields, a case which was considered in [3].

In this limit the two solutions are

$$k^2 + 11\rho k \cdot f \cdot f^* \cdot k = \pm 3\rho k \cdot f \cdot f^* \cdot k \tag{12a}$$

$$A = \pm 1. \tag{12b}$$

The two (linear) polarisations are, from equation (9),

$$\varepsilon^{1\mu} = k_\nu F_{\mathrm{L}}^{\nu\mu} \qquad \varepsilon^{2\mu} = k_\nu \tilde{F}_{\mathrm{L}}^{\nu\mu}. \tag{13}$$

Upon making a guage transformation, so that $\varepsilon^{\pm 0} = 0$, these become

$$\boldsymbol{\varepsilon}^1 = -\omega \boldsymbol{E} + (\boldsymbol{B} \times \boldsymbol{k}) + k_x \boldsymbol{k}/\omega \qquad \boldsymbol{\varepsilon}^2 = -\omega \boldsymbol{B} - (\boldsymbol{E} \times \boldsymbol{k}) + k_x \boldsymbol{k}/\omega.$$

Here we have chosen F_{L} as in equation (1) with $p = 0$. The index of refraction, n, for each polarisation, defined by $\omega n = |\boldsymbol{k}|$, is

$$n = 1 + \binom{4}{7}(2\alpha/45\pi)(eE_0/m^2)\sin^4(\theta/2) \tag{14}$$

where θ is the angle between the propagation direction of the laser and external wavefields. For $\boldsymbol{k} \propto -z$, $\theta = \pi$, i.e. the external plane wave travelling antiparallel to the laser, these reduce to:

$$\boldsymbol{\varepsilon}^1 \propto \boldsymbol{E} \qquad \boldsymbol{\varepsilon}^2 \propto \boldsymbol{B}.$$

The maximum effect occurs for antiparallel momenta and perpendicular polarisations. These results were obtained previously for constant crossed fields [3].

The other limiting case that we will consider is

$$k \cdot p \gg \rho k \cdot f \cdot f^* \cdot k$$

the limit of high frequency, compared to the intensity. (Of couse, both the frequency and the intensity must be small compared to m for the Euler-Heisenberg approximation, which we are using, to be valid.) In this limit we obtain a single solution of equation (3), for each sign:

$$k^2 + 11\rho k \cdot f \cdot f^* \cdot k = 0 \tag{15a}$$

$$A^\pm = 3\rho k \cdot f \cdot f^* \cdot k/(\pm 4\rho k \cdot p) \qquad |A^\pm| \ll 1. \tag{15b}$$

Equation ($15a$) implies that the index of refraction is the average of the two obtained in the opposite limit $p \to 0$:

$$n = 1 + (11\alpha/45\pi)(eE_0/m^2)\sin^4(\theta/2). \tag{16}$$

Effectively the laser polarisation is rotating so quickly that the external photon averages the square of the laser field. Again specialising to the case where the external photon is travelling antiparallel to the laser, the two polarisation vectors, $k_\mu f^{\mu\nu}$ and $k_\mu \tilde{f}^{\mu\nu} = k_\mu f^{*\mu\nu}$, correspond to circular polarisations with spin respectively parallel or antiparallel to the laser photons. Thus, when the spin of the external photon is parallel to that of the laser photon, there is a small amplitude for the external photon to emit two laser photons and flip its spin to conserve angular momentum. In the other case, when the spin of the external photon is antiparallel to that of the laser photon, there is a small amplitude for adsorption of two laser photons with an accompanying spin flip.

The index of refraction was evaluated [5] without use of the Euler–Heisenberg approximation, from the graph of figure 1. It was found from the numerical results [5], that

$$n-1 \approx 7.8\times10^{-4}\times 32\pi\alpha(eE_0/m^2)^2$$

at low frequencies and intensities. This agrees with our equation (16) to better than 1%, providing a consistency check on both the numerical and the analytic calculations. The low-frequency and weak-field conditions for the validity of the Euler–Heisenberg approximation are $k\cdot p/m^2 \ll 1$, $e^2 k\cdot f\cdot f^*\cdot k/m^6 \ll 1$. (Note that these are the only two gauge and Lorentz invariants that can be constructed from a plane-wave field with $p^2=0$ and $k^2=0$, in lowest order.) It was found from the numerical evaluation [5] that equation (15*a*) was a fair approximation for

$$0.1 < \sqrt{k\cdot f\cdot f^*\cdot k}/k\cdot p \approx eE_0/m\omega_{\mathrm{L}} < 1$$

$$2p\cdot k/m^2 \approx 4\omega\omega_{\mathrm{L}}/m^2 \leqslant 1.$$

We now briefly consider the case of a linearly polarised laser. The laser field can still be written as in equation (1) but now the electric and magnetic field amplitudes are real

$$e_i = E_0(1,0,0) \qquad b_i = E_0(0,1,0).$$

We again seek a solution of equation (2*a*) by expanding F_{ex} in Fourier modes, $\exp[-\mathrm{i}(k+2np)\cdot x]$, but now we find that it is necessary to keep an infinite number of modes, with arbitrary n. The fact that only two modes occurred for circular polarisation was essentially a result of angular momentum conservation, together with the low-frequency approximation inherent in our Euler–Heisenberg approach: the external photon travelling antiparallel to the laser could not adsorb or emit more than two laser photons without violating angular momentum conservation, in an s wave state. No such constraint arises for linear polarisation since it corresponds to a superposition of both photon spin states. We now find that all Fourier modes have the same polarisation, either $k_\mu f^{\mu\nu}$ or $k_\mu \tilde{f}^{\mu\nu}$, but an infinite set of coupled equations arises for the amplitudes of the different Fourier modes. In the low-frequency limit, we get the same result as before, corresponding to constant crossed fields. In the opposite limit, all higher Fourier modes (with $n\neq 0$) are small, and to lowest order we can replace $F_{\mathrm{L}\mu\nu}F_{\mathrm{L}\lambda\rho}$ in equation (2*a*) by its value averaged over one period, and similarly for $F_{\mathrm{L}}\tilde{F}_{\mathrm{L}}$. The equation then becomes the same as for constant crossed fields, except that the constant field squared is replaced by the *average* of the field squared. Thus we obtain the results of equation (14) except that E_0^2 is replaced by $\frac{1}{2}E_0^2$. When the laser field is changing rapidly, the external photon is only sensitive to its average square value.

Let us now consider the experimental situation. The proposed experiments [1] would have electric fields of $E_0 = 8.1\times10^{12}\ \mathrm{V\,m^{-1}}$, an optical laser with a frequency of

$\omega_L = 4$ eV and external photons with a maximum frequency of about $\omega = 37$ GeV. Thus

$$4k \cdot p/m^2 \approx 8\omega\omega_L/m^2 \approx 4.5$$

$$e^2 k \cdot f \cdot f^* \cdot k/m^2 \approx (eE_0/m^2)^2(\omega/m)^2 \approx 0.195$$

$$\rho k \cdot f \cdot f^* \cdot k/m^2 \approx (2\alpha/45\pi)(eE_0/m^2)^2(\omega/m)^2 \approx 2.0 \times 10^{-5}.$$

Based on the comparison with the numerical results mentioned above, we might expect the Euler-Heisenberg approximation to be reasonably good, even for the maximum frequency. The condition for the frequency to be high compared to the intensity: $\rho k \cdot f \cdot f^* \cdot k \ll 4k \cdot p$ is extremely well satisfied. Assuming a circularly polarised laser, the maximum index of refraction from equation (16) is:

$$n - 1 = (11\alpha/45\pi)(eE_0/m^2)^2 = 2.1 \times 10^{-14}.$$

The amplitude for adsorption (or emission) of a pair of laser photons, with an accompanying spin flip is, from equation (15*b*):

$$A = 3\rho k \cdot f \cdot f^* \cdot k/4k \cdot p = 1.3 \times 10^{-5}.$$

These effects are both extremely tiny at the proposed intensities and frequencies. With an increase in laser intensity by two or three orders of magnitude, they would become observable.

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